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Introduction to Property Theory

The Fundamental Theorems

David Ellerman

This paper inaugurates the mathematical treatment of property theory, proving the two fundamental theorems for the property system that correspond to the two fundamental theorems for the competitive price system.

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Summary findings

The market system consists of a price mechanism built on the foundation of a system of property and contract. In many developing and transition economies the market system functions poorly. In many cases, if not most, the malfunctioning is not simply in the price system (for example, anti-competitive activities) but in the underlying property system (such as contracts being breached, and externalities in the sense of transfers not covered by contracts).

Economic theory tends to take the functioning of the system of property and contract for granted and focuses on the operation of the price mechanism. Property theory focuses on that underlying system of property and contract.

In this paper Ellerman inaugurates the mathematical treatment of property theory. In contrast with earlier work in "law and economics" and the "new institutional economics," this approach uses principles drawn from jurisprudence and does not attempt to reduce "law" to "economics" in the sense of efficiency considerations such as the minimization of transactions costs. The main results are the two fundamental theorems of property theory that are analogous to the two fundamental theorems of price theory that, in essence, state that:

- A competitive equilibrium is Pareto optimal.
- Given a Pareto optimal state, there exists a set of prices such that a competitive equilibrium at those prices would realize that Pareto optimal state.

This paper—a product of the Office of the Senior Vice President, Development Economics—is part of a larger effort in the Bank to understand the institutional basis for a private property market economy. Copies of the paper are available free from the World Bank, 1818 H Street NW, Washington, DC 20433. Please contact Beza Mekuria, room MC4-358, telephone 202-458-2756, fax 202-522-1158, email address bmekuria@worldbank.org. Policy Research Working Papers are also posted on the Web at http://econ.worldbank.org. The author may be contacted at dellerman@worldbank.org. October 2001. (26 pages)

Introduction to Property Theory:
The Fundamental Theorems

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> We have now run over the three fundamental laws of nature, *that of the stability of possession, of its transference by consent,* and *of the performance of promises.* 'Tis on the strict observance of those three laws, that the peace and security of human society entirely depend; nor is there any possibility of establishing a good correspondence among men, where these are neglected.
>
> David Hume 1739

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* The findings, interpretations, and conclusions expressed in this paper are entirely those of the author and should not be attributed in any manner to the World Bank, to its affiliated organizations, or to the members of its Board of Directors or the countries they represent.

Introduction

The market system consists of the price mechanism built upon the foundation of the system of private property and voluntary contracts. In the economic analysis of the price mechanism ("price theory"), the appropriate operation of the underlying property system is assumed. Here that assumption is relaxed to focus the analysis on the operation of the system of property and contracts ("property theory"). In many transition and developing countries, the market system as a whole does not function well. In many—if not most—cases, the failure is not simply in the price mechanism but in the underlying system of property and contract.

The main attributes of the (competitive) price system are demonstrated in two fundamental theorems that relate competitive equilibria to Pareto optimal (or allocatively efficient) allocations. The theorems show that (1) under certain assumptions, a competitive equilibrium is Pareto optimal, and that (2) under certain assumptions, given a Pareto optimal state, there exists a set of prices such that a competitive equilibrium at those prices will realize the Pareto optimal state.[1] My purpose here is to develop the mathematical framework of property theory in a simple (one-period flow) model and to prove the two theorems for the property system that correspond to the two fundamental theorems about the competitive price system. These results inaugurate the field of mathematical jurisprudence in its own right—where, unlike the work in the field of law and economics or in most of the new institutional economics, there is no attempted reduction of the jurisprudential principles to efficiency considerations such as the minimization of transactions costs.

It is sometimes noted that human freedom is an end in itself, not just a means to achieve efficiency. A decentralized private property system has virtues that are much more robust than the highly special conditions of a competitive equilibrium[2]—such as the realization of people's intentional decisions with social coordination and mutual adjustment carried out by voluntary contracts and such as the principle of reaping the fruits of one's labor. These broader virtues are attributes of the system of private property and voluntary contracts and have been emphasized by economist-philosophers such as John Locke, David Hume, Adam Smith, Frank Knight, Michael Polanyi, and Friedrich Hayek. They are part of the subject matter of property theory.

Graph Theoretical Preliminaries

Directed Graphs

Intuitively, a "directed graph" has a finite set of points or nodes with arrows or directed arcs between them.

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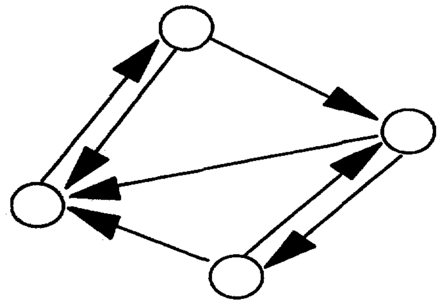
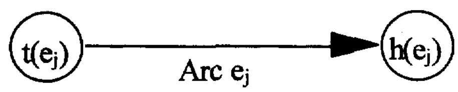
[1] See Arrow 1951, Debreu 1951, Gale 1955, or Quirk and Saposnik 1968.

[2] The work of Stiglitz [see 1994 for a summary] and colleagues in information economics shows that these assumptions are even more special and less robust than previously thought.



A Directed Graph

In more technical terms, *directed graph* $G = (G_0, G_1, t, h)$ is given by a set G_0 of *nodes* (numbered $i = 1, \ldots, I$), a set G_1 of *arcs* (numbered $j = 1, 2, \ldots, J$), and *head* and *tail functions* $h, t: G_1 \rightarrow G_0$ which indicate that arc e_j is directed from its tail, the $t(e_j)$ node, to its head, the $h(e_j)$ node.

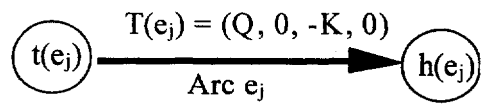


Each node represents a group or party interpreted always as a natural person or group of persons acting together as a unit, and the arcs between nodes represent the "channels" for transfers between the parties.

Assignments to Arcs and Nodes

Real n-dimensional vectors[3] (each component representing a type of commodity including money as a component) will be assigned to the arcs to represent transfers between parties. A function $T: G_1 \rightarrow \mathbf{R}^n$ is a *arc assignment* indicating that, given an arc (or "edge") e_j, the vector $T(e_j)$ is transferred from the $t(e_j)$ party (on the tail of the arrow) to the $h(e_j)$ party (on the head of the arrow) with negative components representing transfers in the opposite direction. For instance, if $T(e_j) = (Q, 0, -K, 0)$, then Q units of the 1st commodity are transferred from the $t(e_j)$ party to the $h(e_j)$ party with K units of the third commodity transferred in the opposite direction.

[3] There is an alternative to using n-dimensional real vectors with positive and negative components. For over five centuries, double-entry bookkeeping has been used to describe the stocks and flows of scalar notions of value within and between parties. In earlier work two decades ago to develop mathematical machinery for property theory, double-entry bookkeeping was mathematically formulated for the first time (!) and then generalized to n dimensions to describe the stocks and flows of physical quantities that underlie conventional scalar accounting [see Ellerman 1982, 1986, or 1995]. The key idea was that the "T-accounts" of double-entry accounting were the ordered pairs in the group of differences construction which extends the monoid of non-negative numbers or even non-negative n-vectors, which have no additive inverses, to an additive group with additive inverses. The same ordered-pairs trick is used multiplicatively to extend the multiplicative monoid of integers to the rationals which include multiplicative inverses (division) and where the ordered pairs are written vertically as "fractions." Double-entry bookkeeping is mathematically formulated and extended to n dimensions by using (horizontally written) ordered pairs or "T-accounts" where the debit and credit entries are non-negative n-vectors. Property theory could be developed using that machinery of n-dimensional double-entry accounting. However, the additive group of T-accounts of non-negative n-vectors is isomorphic to the additive group of $\mathbf{R}^n$ so property theory will be developed simply using $\mathbf{R}^n$ with vectors of both positive and negative components.



In due course, various types of transfers will be considered (e.g., voluntary or not, intended or actual) but for now the focus is on the underlying mathematical machinery.

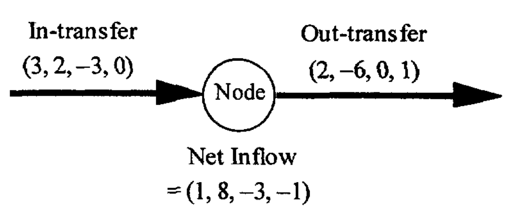
A *node assignment* S: $G_0 \rightarrow \mathbf{R}^n$ assigns real n-dimensional vectors to the nodes to represent net flows within the parties represented by the nodes. Note well that since a node is not "directed," there is no standard interpretation of the signs in the components of the vectors assigned to nodes. Components of one sign represent commodities consumed, used up, or added to stock, and the components of the other sign represent commodities produced or subtracted from stocks during the time period.

There is no need to segregate the parties, represented by nodes, into "producers" and "consumers"; all parties are capable of both activities. If the value of S at node n_i is $S(n_i) = (Q,0,-K,0)$, then it might mean that the i^{th} party produced Q (or removed it from previous stocks) and used up or consumed K (or added it to stocks) during the time period. If S represented production activities as positive components with consumption as the negative components, then –S would represent the consumption activities as positive with production as the negative components.

The Net Inflow or Boundary Operator

Given an arc assignment of transfers T, each node will have certain property vectors transferred into it and out from it. The net in-transfer at each node can then be computed, and it would be a node assignment ∂T: $G_0 \rightarrow \mathbf{R}^n$ determined by the given arc assignment T. If (3, 2, –3, 0) was transferred into a node and (2, –6,0, 1) was transferred out of the node, then the net in-transfer or *net inflow* is:

$$(3, 2, -3, 0) - (2, -6, 0, 1) = (1, 8, -3, -1).$$



Given the arc assignment T, the node assignment ∂T is defined at a node n_i by adding all the $T(e_j)$ where the arc e_j has node n_i at the head of the directed arc, and subtracting $T(e_j)$ where arc e_j has node n_i at the tail of the arc. In technical terms, the node assignment ∂T is defined as:

$$\partial T(n_i) = \sum_{h(e_j)=n_i} T(e_j) - \sum_{t(e_j)=n_i} T(e_j)$$

for nodes $n_i = n_1,\ldots,n_I$. The node assignment ∂T is called the *net inflow* or, in mathematics, *boundary* of the arc assignment T [e.g., Aigner 1979, 358; Giblin 1977, 27; or "divergence" with the opposite sign convention in Rockafellar 1984, 11]. Thus ∂ is the *boundary operator* that carries arc assignments T into node assignments ∂T. If a commodity is both transferred to and away from a party during the time period, then it will cancel out in the calculation of the net inflow at that node.

Basic Mathematical Results

Chain Groups

Let $C(G_1, \mathbf{R}^n)$ be the set of arc assignments and let $C(G_0, \mathbf{R}^n)$ be the set of node assignments. Each will be taken as an additive group by taking $(T + T')(e) = T(e) + T'(e)$, and similarly for nodes. It is sometimes convenient to write the members of these groups as formal sums called "chains" and then the groups are "chain groups." For instance if the arcs are e_j for $j = 1,\ldots,J$ then the arc assignment T could be written as the formal sum:

$$T(e_1)e_1 + T(e_2)e_2 + \ldots T(e_J)e_J$$

called a *1-chain* [e.g., Giblin 1977, 26]. The formal sums in $C(G_0, \mathbf{R}^n)$ are called *0-chains*. The boundary or net inflow function defined by

$$\partial T(n_i) = \sum_{h(e_j)=n_i} T(e_j) - \sum_{t(e_j)=n_i} T(e_j)$$

is a group homomorphism:

$$\partial: C(G_1, \mathbf{R}^n) \rightarrow C(G_0, \mathbf{R}^n).$$

Given a node assignment S, its *augmentation* Aug(S) is the sum in $\mathbf{R}^n$ of values assigned to all the nodes n_i for $i = 1,\ldots,I$:

$$\mathrm{Aug}(S) = \sum_{i=1}^{I} S(n_i)$$

which gives a group homomorphism: Aug: $C(G_0, \mathbf{R}^n) \rightarrow \mathbf{R}^n$.

The Augmentation of any Boundary is Zero

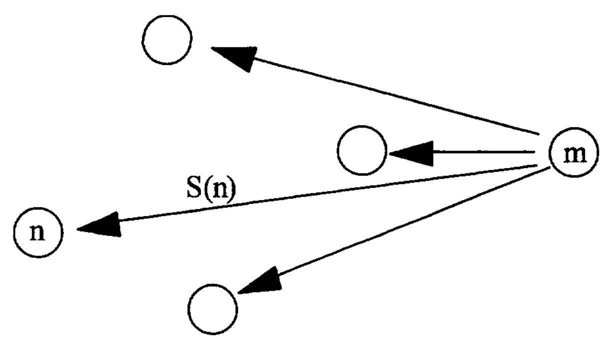
In the boundary ∂T of any arc assignment T, each value T(e) on an arc e occurs exactly once positively at the node on the head of the arc and exactly once negatively at the node on the tail of the arc. Thus the augmentation of any boundary is zero (see Appendix 1 for the proof).

<u>Theorem 1</u>: For any T in $C(G_1, \mathbf{R}^n)$, $\mathrm{Aug}(\partial T) = 0$.

The *image* $\mathrm{Im}(\partial)$ of the boundary homomorphism is a subgroup of $C(G_0, \mathbf{R}^n)$ consisting of all the node assignments that are the boundary of some arc assignment. The *kernel* Ker(Aug) of the augmentation is the subgroup of all the node assignments mapped to 0. Theorem 1 then says that the image of the boundary is contained in the kernel of the augmentation, $\mathrm{Im}(\partial) \subseteq \mathrm{Ker}(\mathrm{Aug})$.

The Associated Market Graph

The economic motivation of some of the concepts will become clearer if one associates with each graph G, the *associated market graph* G_m with the "market" added as an extra (fictitious) node m, with an arc added directed from the market node m to each node n of G, and with the original arcs deleted (so each party relates only to the market).



Associated Market Graph G_m

With each node assignment S on G, there is an associated arc assignment T_S on G_m where S(n) is assigned to the arc from the market node m to the node n and clearly any arc assignment on G_m could be associated with a unique node assignment on the original graph. Indeed, the chain group of arc assignments on G_m is isomorphic to the chain group of node assignments on G. Then the net inflow or boundary of T_S at m is minus the augmentation of the node assignment S,

$$\partial T_S(m) = -\mathrm{Aug}(S),$$

and the sum of the boundaries of T_S over all the original nodes of G is the augmentation of S:

$$\sum_{i=1}^{I} \partial T_S(n_i) = \mathrm{Aug}(S).$$

The augmentation of any boundary is zero (Theorem 1), and, in the case of T_S, it has the special form: $\mathrm{Aug}(\partial T_S) = \mathrm{Aug}(S) - \mathrm{Aug}(S) = 0$. Taking the augmentation of a node assignment on the original graph is equivalent to computing the "quantity balance" or net outflow at the market node in the augmented graph. Taking the positive components in an arc assignment from the market node m to any other node n as the quantity demanded by the party n from the market and negative components as the quantity supplied by n to the market, then the overall excess supply would be the net inflow $\partial T_S(m) = -\mathrm{Aug}(S)$ at m.

Now the idea is to use the market-style reasoning of the associated market graph to approach Theorem 2 : given a node assignment S satisfying the quantity-balance condition Aug(S) = 0, there exists an arc assignment T such that $\partial T = S$. This proposed Theorem 2 states that $\mathrm{Ker(Aug)} \subseteq \mathrm{Im}(\partial)$ so it is the converse to Theorem 1. The strategy is to show how the theorem is trivial on the associated market graph and then to get the general proof by carrying over that market-style reasoning to the original graph G.

Given an arbitrary node assignment S on G, it is trivially extended to a node assignment S* on the node set of G_m by assigning S*(m) = –Aug(S) and otherwise S*(n) = S(n). The extra node m serves as the "market-maker" by absorbing the excess supply from S [i.e., S*(m) = –Aug(S)], so (by construction) Aug(S*) = 0 and T_S is an arc assignment on the associated market graph such that $\partial T_S = S^*$.

Any Node Assignment with Zero Augmentation is a Boundary

By reproducing this market-maker reasoning *within* the original graph G, I arrive at the proof of Theorem 2. The proof is the customary one in homological algebra [e.g., Giblin 1977, pp. 31-2] which seems to be arrived at without any knowledge of the underlying market interpretation. Instead of the extra market node m, I select arbitrarily any node n_m in G to play the role of "market-maker" to absorb whatever is the excess supply from the other nodes. Instead of constructing a node assignment S* with zero augmentation, I start with an assignment S with Aug(S) = 0. Instead of the arcs constructed to any node n from m, I need the notion of a "path" to any other node n from the market-maker node n_m. That requires some definitions.

Given nodes n and n*, a *path from* n *to* n* is a sequence of arcs ee'e"...e* such that the head or tail of e (respectively e*) is the node n (resp. n*), and where each two consecutive arcs in the path share a common node. An arc occurs *positively* (resp. *negatively*) along the path if the direction of the arc (as an "arrow") is along (resp. against) the direction of the path from n to n*. Clearly each path can be identified with a 1-chain ±e±e'±e"...±e* in $C(G_1, \mathbf{R}^n)$ where the coefficient on each arc is determined by whether the directed arc lies positively or negatively along the path. The negative of a path from n to n* is a path from n* to n.

A graph is *connected* is there is a path between any two nodes. G is henceforth assumed to be connected.

<u>Theorem 2:</u> Let S be a node assignment on a connected graph G such that Aug(S) = 0. There exists an arc assignment T on G such that $\partial T = S$, i.e., $\mathrm{Ker(Aug)} \subseteq \mathrm{Im}(\partial)$ [see Appendix 1 for a proof].

Theorems 1 and 2 together imply that $\mathrm{Ker(Aug)} = \mathrm{Im}(\partial)$. In the language of homological algebra, the sequence

$$C(G_1,\mathbf{R}) \xrightarrow{\partial} C(G_0,\mathbf{R}) \xrightarrow{\mathrm{Aug}} \mathbf{R}$$

is *exact*. Theorem 2 says that given a set of production-consumption intentions for each party such that the quantities balance or "market clears" overall, there exists a set of bilateral exchanges along the given channels connecting the parties that would support the production-consumption decisions.

Theorem 2 can also be given a more indirect proof (without the market-maker motivation) using linear algebra.[4] The theorem can be generalized by adding upper and lower capacities to the arcs and then accordingly strengthening the existence conditions. The generalized existence results were proven by David Gale [1957] and A. J. Hoffman [1960], and were stated in a clear general form by Rockafellar as the Gale-Hoffman "Feasible Distribution Theorem" [1984, p. 71].

Introductory Property Theory

Preliminaries

Both the property system and the price system (which operates on top of the property system) are concerned with the general problem of socially coordinating the individual decisions of the parties. Competitive price theory is a theory about how a decentralized system of competitive markets can carry out this function in a certain optimal or efficient manner. The decentralized system of property and contract is concerned not with optimization but with the more rudimentary matter of voluntary and intentional actions; it uses the system of voluntary contracts as the mutual-adjusting and socially coordinative mechanism. The theory has a "quantity" flavor since prices (equilibrium or disequilibrium, competitive or not) are only relevant in determining the quantities of money or other commodities that might be traded in a voluntary exchange.

Price theory models the parties as making decisions or, at least, determining intentions by optimizing some objective function within feasibility constraints. But for the purposes of the property theory theorems, we can take intentions and consents as primary data. The optimizing (or satisficing) of objectives and the satisfaction of the feasibility constraints is all in the background and need not be explicitly considered for property-theoretic purposes.

A property system is concerned with inter-party interactions and ignores the unintended effects on one party not caused by other parties. It is a concern for the property system if a farmer's crop is stolen, but not if the crop is lost in a storm. The difference is that another party was involved in a theft, not in storm damage. Theft, fraud, and coercion are involuntary or unintentional transfers of the possession of commodities between parties. The concept of factual "possession" is constantly being redefined by the laws and cases in a legal system. In order to emphasize the inter-party aspect, once a commodity comes into the possession of a party, it stays that way until it comes into the possession of another party (even though the possessor may not be physically using the commodity).

Given the intended actions of the various parties, one way to connect and coordinate them would be to have a bureaucracy operating a central clearing house—the "socialist" solution. Indeed, that centralized approach is used *within* most large organizations.[5] A system of legal contracts

[4] See Slepian 1968, Theorem 3, p. 92, or Strang 1986, Box 1Q, p. 74 for a clear treatment.

[5] Some large organizations are experimenting with market-like contracting mechanisms between subunits where the decentralized decisions are otherwise assured or assumed to be in line with organizational goals, e.g., Pinchot and Pinchot 1993, Ackoff 1994, or Halal 1996. The property theoretic theorems could also be applied, *mutatis mutandis*, to this situation within a large organization.

and property rights is a decentralized alternative (which still requires the operation of centralized legal authorities) that emphasizes the decision-making autonomy of the parties. Property theory shows how that system addresses the basic problem of realizing the compossible intentions of the various parties.

Price theory tends to ignore the actual bilateral or multilateral structure of exchanges by treating each party as dealing with the market (after the fashion of the associated market graphs considered above). This treatment of market transactions as if they all went through a central clearing house has inspired various socialist attempts to "mimic the market." Property theory restores the focus on decentralized transactions directly between the parties. Even a purchase from a vending machine (the standard example of an impersonal transaction with "the market") is actually a bilateral exchange between the customer and the operator of the vending machine. Hence the theory uses general connected graphs to model transactions (instead of the particular type of graph seen in the associated market graphs). Instead of just considering the node vectors of the parties (which was sufficient on an associated market graph), the transfers between parties must also be included. The transfers in an arc assignment T are said to *support* the actions given by a node assignment S if $\partial T = S$.

In the fashion of introductory treatments of consumption and production in price theory, a one-period flow model is used. Assignments to arcs and nodes will represent flows of commodities during the time period. "Commodities" refers to separable private goods that can be transferred between parties, not goods with public or jointness attributes where decentralized mechanisms need to be supplemented by other social choice mechanisms.[6] When a vector is assigned to a directed arc, the positive components represent transfers of those commodities along the arc (as a "trade channel" between the parties represented by the nodes) while the negative components represent transfers against the direction of the arc. Flows at the nodes result from either production-consumption activities or from changes in stocks or inventories made by the parties during the time period. Nodes are undirected so the interpretation of the positive and negative components is determined by the context.

Normative Concepts of Property Theory: Initial Treatment

The basic normative and institution-free (e.g., price-free) notion in price theory is that of Pareto optimality (or allocative efficiency). The basic institutional notion in price theory is that of a competitive equilibrium in the price system. The "Fundamental Theorems of Welfare Economics" show in what sense the price system is efficient by relating the notions of a competitive equilibrium and Pareto optimality. The first theorem is that (1) under certain assumptions, a competitive equilibrium is Pareto optimal; and the second theorem is that (2) given a Pareto optimal social state, there exists, under certain conditions, a set of prices such that a competitive equilibrium at those prices yields or supports that Pareto optimal state.

In order to construe Pareto optimality in our simple model, a utility or welfare function would be associated with each party (node) and it would be a function of the node assignment. A Pareto optimal node assignment would be a feasible one such that no party's welfare function could be

[6] See, for example, Frank Knight's treatment of government by discussion [1956, 266-7].

increased by another feasible assignment without making another party worse off. Each party would be at a maximum that is compossible with the maximum achieved by the other parties.

Underlying optimizing behavior is the broader notion of intentional action (which is taken here as a primitive notion). If one ignores the optimization aspect in favor of simply considering a set of intentional actions by the parties, then the compossibility or social consistency requirement reduces down to the quantity balance condition, i.e., that the augmentation of the node assignment representing intended actions is zero. Thus to find the property-theoretic analogue of the notion of "Pareto optimality," intentional action replaces optimizing action and the quantity balance condition replaces the compossibility condition on the optimizing of the different parties.

A node assignment is said to be a *social intentional action* if it represents intentional decisions of the parties that satisfy the "social" or compossibility condition of a quantity balance (i.e., has zero augmentation). "Intentional" is stronger than and entails "voluntariness." In realizing a social intentional action, each party is not just passively consenting to how it is affected but is carrying out an intentional or deliberate plan of action and is *de facto* responsible for the results.

Let SIA, as in "social intentional action," stand for any node assignment representing intentional production-consumption and inventory-change decisions or plans of the parties that satisfy the quantity balance condition, i.e., Aug(SIA) = 0. The sign conventions are chosen so that the positive components of SIA at a node represent the commodities the party intends to consume or add to stocks while the negative components represent the commodities the party intends to produce or subtract from stocks. The negative –SIA describes the same socially consistent plans but with the opposite sign convention (i.e., with production as positive and consumption as negative).

The Scottish Enlightenment (e.g., Hume and Smith) and the later Austrian literature in economics have emphasized the role of markets to coordinate free activity as an end in itself, not simply as a means to achieve an efficient allocation of resources. The property-theoretic theorems given below (initial formulation) could be seen as an attempt to formally render *that* role of a private property market economy. More recently, Sen [1999] has emphasized that while freedom may play an instrumental role in reaching a Pareto optimal state, it also has a constitutive role. The notion of a social intentional action and the property theorems show in a simple way how the property system realizes this constitutive role of freedom, the realization of people's socially consistent intentional and responsible actions.

The Contractual Mechanism of the Property System

Property theory is concerned with how a decentralized system of voluntary exchange will realize social intentional actions that are compossible in the sense of the quantity balance condition just as price theory is concerned with how a decentralized system of competitive markets will realize the compossible optimizations of a Pareto optimal (or "allocatively efficient") state.

The contractual mechanism is an institutional means to deal with the transfers of physical quantities of commodities between the parties. Formally the idea is quite simple; enforce certain types of transfers (arc assignments) whose boundary is intentional. Intuitively, these transfers are the mutually voluntary contracts. The possibility of fraud renders the notion of a "voluntary" transfer as more complicated than simply a transfer accompanied by "consent." A party might consent to pay $10 for a certain commodity but unknowingly receive a worthless substitute. The transaction has the appearance of voluntariness but the consent was to a rather different transaction. Thus "mutual voluntariness" is always intended in the strong sense of a knowing "meeting of the minds" about the actual commodities exchanged.

Legal transfers or *contracts* are defined as being mutually voluntary transfers. Let VC, as in "voluntary contracts" (in spite of the redundancy since all contracts are voluntary), stand for any arc assignment representing mutually voluntary transfers of commodities between the parties. In view of the interpretation, it is reasonable to assume that the boundary of voluntary contracts is a set of intentional decisions by the parties. If the party at a node voluntarily agrees to each of the transfers in VC in which the party is involved, then surely the net inflow ∂VC at that node represents the party's intentional decisions SIA. In any case, the assumption ∂VC = SIA is made. From the strictly formal viewpoint, it only matters that the contractual mechanism enforces the arc assignments which have a certain attribute ("voluntariness") whose boundary has the attribute desired of node assignments ("intentionality").

The transfers actually carried out might diverge from the expectations based on mutually voluntary contracts. A farmer might intentionally use inputs to produce some outputs but part of the outputs might be taken by another party involuntarily or the farmer might have stolen or "converted" some of the inputs from another party. Let FT, as in "factual transfers," be the arc assignment representing the transfers in the possession and control of commodities that actually takes place (involuntarily or voluntarily) between the parties during the period. This means that ∂FT represents the actual or realized net inflows at the nodes. How will the contractual mechanism lead to the realized node assignment ∂FT being a social intentional action?

The "equilibrium" in the contractual mechanism, FT = VC, is when all factual transfers of commodities are mutually voluntarily and all mutually voluntary agreements are fulfilled by the corresponding factual transfers. It would be misleading to take VC as "given" data to be compared to the "given" actual transfers FT. The contractual mechanism is a negative feedback system that operates to eliminate any difference between VC and FT. There are two ways this matching is brought about by the contractual mechanism.

Given a factual transfer, the contractual mechanism works to insure that it is covered by or can be construed as a voluntary contract (i.e., works to eliminate property externalities) by allowing legal recourse for damages to any aggrieved party (e.g., an alleged victim of a theft or conversion). The factual transfers not covered by voluntary contracts are called *externalities*.

And given an intended voluntary transfer representing a meeting of the minds, i.e., a contract, the contractual mechanism will work to insure that the contract is fulfilled by the indicated transfer in the possession and control of the commodities between the parties. This is typically done by allowing legal recourse for damages to one party if that party has fulfilled its part of the contract but the other party has allegedly breached its part. The legal contracts unfulfilled by factual transfers are called *breaches* [see Appendix 2 for more analysis of externalities and breaches].

This negative feedback *contractual mechanism* aims at the "equilibrium" FT = VC which will be called a *contractual matching* since it means the actual or realized transfers match the intended mutually voluntary transfers of commodities. It should be noted that the price theory theorems already assume that this contractual mechanism has done its job to achieve a matching. That is, the price-theoretic fundamental theorems assume that there are no property externalities (no extra-contractual transfers) and that the contracts determined by optimizing decisions are fulfilled. Our concerns are precisely those prior questions of insuring that actual transfers in possession are covered by contracts (no externalities) and that contracts are fulfilled by actual transfers.

For the moment, I have avoided any mention of "property rights" so they can be introduced later. The initial formulation of the two fundamental property theorems deal with the case where the actual transfers match the contractual transfers, and the contractual mechanism can be described in terms of insuring that actual transfers of commodities are covered by contracts and that contracts are fulfilled by actual transfers. For the next two theorems, it is not necessary to introduce "property rights" nor to interpret the contracts as transferring property rights. The theorems will be later restated in those terms.

The Fundamental Theorems of Property Theory: First Formulation

What is the property theoretic analogue to the price theoretic theorem that a competitive equilibrium is Pareto optimal? It is the rather straightforward result that if there is a contractual matching, then a social intentional action is achieved.

Theorem 3: First Fundamental Theorem (Initial Formulation): If there is a contractual matching, then the actions realized by the parties constitute a social intentional action.

Proof. With VC = FT, the realized actions ∂FT are intentional since ∂FT = ∂VC (= SIA) and satisfy the quantity balance condition by Theorem 1. QED

Thus if the contractual mechanism does its job to insure that all actual transfers are mutually voluntary and that all intended mutually voluntary contracts are fulfilled by the indicated transfers, then the decisions realized by the parties are intentional and socially consistent. By the contractual mechanism, the parties are led, as if by an invisible hand, to realize their intentions in a socially compossible way.

The "converse" theorem will investigate when a given social intentional action of the parties can be supported and realized by the contractual system of voluntary transfers.

Theorem 4: Second Fundamental Theorem (Initial Formulation): Given a set of intended actions SIA which satisfy the quantity balance condition Aug(SIA) = 0, there exists a supporting set of transfers T between the parties such that if T is accepted by the parties as voluntary contracts VC, then the contractual matching would realize the social intentional action SIA.

Proof: The existence of T follows directly from Theorem 2 and the rest is immediate from the definitions. By the existence of T accepted as VC, we have ∂VC = SIA, and the matching VC = FT implies ∂FT = ∂VC = SIA. QED

Although formulated at a high level of abstraction, Theorems 3 and 4 show the simple effectiveness of the decentralized system of private property and contracts to realize the intentions of the various parties so long as those intentions are socially consistent. The contractual mechanism works if prices are in equilibrium or not, if markets are competitive or not, and, indeed, if the exchanges involve money or only barter.

Intellectual Background

The ideas of the contractual mechanism in the property system and the price mechanism in the competitive market system as constituting spontaneous orders or invisible-hand mechanisms were developed during the Scottish Enlightenment principally by David Hume and Adam Smith. Indeed it would seem appropriate to attribute the intellectual description of the contractual mechanism to Hume just as the description of the competitive market system is attributed to Smith. In Hume's *Treatise of Human Nature* (first published in 1739 well before Smith's *Moral Sentiments* in 1759 or *Wealth of Nations* in 1776), he focuses on the operation of the system of property and contract and develops the principles involved in the legal machinery enforcing a contractual matching.

> We have now run over the three fundamental laws of nature, *that of the stability of possession, of its transference by consent,* and *of the performance of promises.* 'Tis on the strict observance of those three laws, that the peace and security of human society entirely depend; nor is there any possibility of establishing a good correspondence among men, where these are neglected. [Hume 1978 (1739), Book III, Part II, Section VI, p. 526]

The "transference by consent" means that factual transfers are covered by voluntary contracts (no externalities), and "performance of promises" means that voluntary contracts are fulfilled by the corresponding factual transfers (no breaches) so, together (by Theorem 7 in Appendix 2), Hume describes a contractual matching FT = VC.

In Michael Polanyi's treatment of the system of property and contract as a spontaneous order, he cites similar rules that Leon Duguit extracted from French jurisprudence.

> In the Code Civil of France (leaving out of account the law of the family) Duguit finds only three fundamental rules and no more—freedom of contract, the inviolability of property, and the duty to compensate another for damages due to one's own fault. Thus it transpires that the main function of the existing spontaneous order of jurisdiction is to govern the spontaneous order of economic life.... No marketing system can function without a legal framework which guarantees adequate proprietary powers and enforces contracts. [Polanyi 1951, 185]

Frank Knight emphasizes the point that the exchange system works with separable private goods so the participants do not have to come to an agreement about a common end (unlike members of a unitary organization who would require government by discussion or some other social decision-making procedures), and he reduces the rules of justice to the essentials.

> The supreme and inestimable merit of the exchange mechanism is that it enables a vast number of people to co-operate in the use of means to achieve ends as far as their interests are mutual, without arguing or in any way agreeing about either the ends or the methods of achieving them. It is the 'obvious and simple system of natural liberty.'... The only agreement called for in market relations is acceptance of the one essentially negative ethical principle, that the units are not to prey upon one another through coercion or fraud. [Knight 1956, 267]

Friedrich Hayek cites both Hume's rules of justice as well as the modern contribution of Duguit [Hayek 1976, 40]. Hayek used the term "catallaxy" to refer to "the special kind of spontaneous order produced by the market through people acting within the rules of the law of property, tort and contract." [Hayek 1976, 109]

> The decisive step which made such peaceful collaboration possible in the absence of concrete common purposes was the adoption of barter or exchange.... All that was required to bring this about was that rules be recognized which determined what belonged to each, and how such property could be transferred by consent. [Hayek 1976, 109]

Hayek goes on to note [p. 185] that it "was these rules to which David Hume and Adam Smith emphatically referred as 'rules of justice'..." In Smith's *The Theory of Moral Sentiments*, Smith argued that society need not depend on "mutual love and affection" or on beneficence.

> Society may subsist among different men, as among different merchants, from a sense of utility, without any mutual love or affection; and though no man in it should owe any obligation, or be bound in gratitude to any other, it may still be upheld by a mercenary exchange of good offices according to an agreed valuation. [Smith 1969 (1759), Part II, Section II, Chapter III, 124]

Hayek argues that even without common purposes or mutual bonds, it is the observance of the rules of disinterested exchange that Smith sees as the "justice" which is

> the main pillar that upholds the whole edifice. If it is removed, the great, the immense fabric of human society, that fabric which, to raise and support, seems, in this world, if I may say so, to have been the peculiar and darling care of nature, must in a moment crumble into atoms. [Smith 1969 (1759), Part II, Section II, Chapter III, 125]

Recently, Charles Lindblom [2001] has explored the essentials of the market system. This system of social coordination and peacekeeping is based on the personal liberty to act and on each party having a protected sphere of property.

> These two rights, liberty and property, seem like perverse foundations for a system of broad social coordination, for they guarantee that, except for specific prohibitions, people can do as they wish. Yet coordination would seem to require that people do not simply do as they wish but instead bend to the requirements of cooperation and peacekeeping. All the more interesting, then, are the ways in which the two rights support cooperation and peacekeeping. [Lindblom 2001, 53]

The third requirement is the quid pro quo that yields voluntary contracts between the parties.

> The three customs and rules [(DE) that is, liberty, property, and quid pro quo voluntary exchanges] create a widespread process of mutual adjustment in which each participant explores innumerable possibilities of benefit for both self and other, thus innumerable opportunities for cooperating and reducing conflict. [Lindblom 2001, 54]

Introducing Property Rights Explicitly

Stock-Flow Considerations

In the factual transfers FT, it is the possession and control of physical quantities of commodities that is transferred. "Property rights" are constructed so that they are what are transferred by the voluntary contracts VC enforced by the legal system. The definition of "property rights" needs to cover the whole "life cycle": how property rights are initiated, transferred, and terminated.

First I must go "inside the black box" at each node to analyze the stocks and flows within a time period. The basic identity is:

$$\text{Ending Stock} = \text{Beginning Stock} + \text{Inflows} - \text{Outflows, or}$$
$$\text{Net Increase in Stock} = \text{Net Inflows.}$$

For the factual flows of physical quantities of commodities, the net inflows from transfers with the other parties are given at each node by the boundary ∂FT of the factual transfers. But there are also increases and decreases due to the production (creation) or consumption (destruction) of commodities. The production or creation of a commodity will be called the *factual appropriation* of the commodity, and the consumption, using up, or otherwise destruction of the commodity would simply be treated as the negative of factual appropriation. Hence let FA', as in "factual appropriation," be the node assignment giving factual creation and destruction of commodities by each party during the time period. The sign conventions are taken so that the positive components of FA' represent commodities produced or otherwise created, while the negative components represent the commodities consumed, used up, or otherwise destroyed (always in net terms) during the time period.[7]

Finally let ΔFH, as in "change in factual holdings," be the node assignment representing the changes in the stocks, inventories, or holdings in the factual possession and control of the parties during the time period (positive components represent increases in holdings and negative

[7] The sign convention to represent appropriation is in accord with the convention for production sets in economics where produced outputs are positive and used-up inputs are negative, e.g., Quirk and Saposnik 1968, 27.

components represent reductions in holdings). Since FA' and ∂FT are two forms of net inflows (from production/consumption within the party and from other parties), the stock-flow identity for the possession and control of physical commodities is: $\Delta FH = FA' + \partial FT$.

The flows in property rights are rendered in the same way with ∂VC being the net transfers in property rights to each party from other parties. The creation of a property right is the "legal appropriation" of the right, and the termination of a legal property right will be treated as the negative of appropriation.[8] Let LA', as in "legal appropriation," be the node assignment with the positive (negative) components representing the initiation (termination) of property rights by the parties during the time period. Finally let ΔLH, as in "change in legal holdings," be the node assignment representing the changes in the existing property rights to commodities already held by the parties during the time period. Then the stock-flow identity for legal property rights is: $\Delta LH = LA' + \partial VC$.

There are two ways to treat the effects of the passage of time on stocks. One way is to treat the "same" stock as carrying over into the next time period and only becoming one period "older." The other way is to treat the carrying over of stocks as the using up of the stocks in the first period and the creation in the next period of one-period-older stocks. It will simplify our formalism and not sacrifice any interpretation if I choose the second method.

Thus the factual equation might be viewed as: $0 = [-\Delta FH + FA'] + \partial FT$, where the term in square brackets is the broadened "production-consumption" term including the "production" of the stocks carried over from the prior period and the "consumption" of the stocks carried over to the next period. Now I broaden the interpretation of factual appropriation to include these stock changes, i.e., $FA = [-\Delta FH + FA']$, so that the factual equation can be simplified to: $0 = FA + \partial FT$. Since the boundary of the factual transfers is the net inflow, the equation could be expressed as the:

<u>Factual Boundary Conditions</u>:

$$FA = -\partial FT$$

"Factual Appropriation" = "Net Factual Outflows."

The same reinterpretation will be applied to the legal flow identity so the legal appropriation $LA = [-\Delta LH + LA']$ includes the negative of the changes in legal holdings. Then the equation would be the:

<u>Legal Boundary Conditions</u>:

$$LA = -\partial VC$$

"Legal Appropriation" = "Negative Boundary of Voluntary Contracts."

[8] Indeed the termination of rights was an original meaning of "expropriation." "This word [expropriation] primarily denotes a voluntary surrender of rights or claims; the act of divesting oneself of that which was previously claimed as one's own, or renouncing it. In this sense, it is the opposite of 'appropriation'. A meaning has been attached to the term, imported from foreign jurisprudence, which makes it synonymous with the exercise of the power of eminent domain, ... " (Black 1968, 692, entry under "Expropriation") Since "expropriation" now has this acquired meaning, I will treat the "expropriation (termination) of rights to the assets +X" as the "appropriation of the liabilities –X."

The Invisible Judge: The Market Mechanism of Legal Appropriation

I have already considered the interpretation of the voluntary contract term VC and the contractual mechanism implemented by the legal authorities. It remains to interpret what I have called the "legal appropriation" term LA which equals $-\partial VC$ by the formal stock-flow identities. What is the legal mechanism, in effect, used by the legal authorities to initiate and terminate property rights, and does it equal $-\partial VC$? This turns out to be an "invisible hand" mechanism of some interest.

Consider, by contrast, an example of the "visible hand" of the legal authorities intervening to legally assign initial or terminal property rights to a party. Suppose goods owned by one party have been destroyed allegedly by another party. The owner takes the other party to court and sues for damages. If no trial or judgment was made, then the owner would "swallow" the costs of the destroyed property and would have, in effect, terminated the legal property right—which could be viewed formally as the "appropriation of the liability" for the property. But when the trial is held, the legal authorities try to ascertain the facts and to assign the legal liability or legal responsibility for destroying the property to the charged party if that party is found to indeed be *de facto* responsible for destroying the property (i.e., Duguit's duty to "compensate another for damages due to one's own fault"). The payment of material damages to the owner, in effect, makes one last transfer of the property right to the guilty party where that property right is terminated.[9] This is expressed in simple terms in the china shop sign: "If you break it, you bought it."

This operation of the visible hand of the legal authorities allows us to conceptualize the "laissez faire" mechanism of appropriation that is, in effect, used when the legal authorities do *not* intervene to make any assignments. The last owner of a commodity that was consumed, used up, or otherwise destroyed is, in effect, the terminal owner (when that party has no legal recourse to get the legal authorities to enforce one last transfer of ownership to some other party). Since the "destruction of a plus" can be expressed mathematically as the "creation of a minus," I will say that the terminal owner of a consumed, used up, or otherwise destroyed commodity *laissez faire appropriates* the liability for that property. This yields a *laissez faire* or market mechanism for the legal termination of property rights, i.e., being the terminal owner of the property rights.

Who is the terminal owner of a property right during the time period? The voluntary contracts transfer ownership, and many parties might buy and sell a commodity during the period. The boundary operator computes the net inflow at the parties, namely the inflows not cancelled by any corresponding outflows during the time period. Hence the positive components of ∂VC at a node gives the property for which that party is the terminal owner. The negative boundary $-\partial VC$ would at each node represent those terminally owned commodities as the negative components. By the legal flow equation, $LA = -\partial VC$, those are precisely the negative components or "liabilities" in the LA node assignment. Thus we see the interpretation of the negative

[9] There may also be punitive damages or fines which function not as a price to make a transaction but as a penalty to enforce the constraint to stay within the market mechanism rather than provoke the intervention of the "visible judge" (e.g., see the different roles of prices and penalties in the "penalty method" in linear programming).

components in LA as representing a *laissez faire* or market mechanism of appropriation of liabilities.

Turn the signs around and we have the market mechanism of appropriation for positive property (or "assets"). Property might be bought and sold many times during the time period but the party who is the initial seller, in effect, appropriates that property right if the legal authorities do not intervene to reassign that property right. This is illustrated in ordinary production. For example, a family farm would purchase or already own the inputs (seed, fertilizer, use of land, and so forth) and these inputs would be used up in production. With no legal action, the farm is the terminal owner of those inputs and thus, by the market mechanism, legally appropriates those liabilities. Then the farm would be the initial seller of the produced outputs. Since the farm had borne the costs of all the inputs ("appropriated the input-liabilities"), no other party would have legal recourse to challenge the farm's initial sale of the produced outputs. In that manner, one could say that the farm *laissez faire appropriated* the liabilities for the used-up inputs and the ownership rights to the produced outputs.[10] Of course, many commodities are both produced and used up within a party but they are of no regard to the property system which is concerned with interparty relationships.

How would those initial sales or legal transfers show up in the formal machinery? Since the boundary of VC computes the net inflows, the negative of that boundary would be the legal outflows not cancelled by any corresponding inflows. Thus the positive components of $-\partial VC$ at a node represent the "first sale" or initial legal transfers of the property rights that are *laissez faire* appropriated by the party at that node during the time period. And by the legal flow equation $LA = -\partial VC$, those are precisely the positive components in the value of LA at the node.

Thus we have arrived at an interpretation of the term LA as expressing the *laissez faire* or market mechanism of appropriation—so the label "legal appropriation" is fitting. The market represented by the voluntary contracts VC acts, in effect, as an "invisible judge" to impute the legal responsibility for the appropriated "assets and liabilities" according to the equation: $LA = -\partial VC$. The appropriation of assets is the beginning of contract, and the appropriation of liabilities follows the end of contract. Hence the laissez faire appropriation principle is:

"Appropriation is the (negative) Boundary of Contract"

("negative" since the boundary computes the net purchases or inflows rather than net sales or outflows). This completes the description of the life cycle of legal property rights: how they are initiated, transferred, and terminated by the market-based system of property and contract.

The *laissez faire* mechanism shows the effects on property rights when the legal authorities do not intervene, i.e., when the law "lets it be." This could include the appropriation of assets from

[10] The closest Hume came to recognizing the legal appropriation of assets in production was his rather inadequate notion of "accession": "We acquire the property of objects by *accession*, when they are connected in an intimate manner with objects that are already our property, and at the same time are inferior to them. Thus the fruits of our garden, the offspring of our cattle, and the work of our slaves, are all of them esteem'd our property, even before possession." [Hume 1978 (1739), Book III, Part II, Section III, p. 509] He did not explicitly recognize the algebraically symmetric appropriation of liabilities.

and the termination of property rights to the commons, e.g., the harvesting of fish from the ocean or nitrogen from the air, or the emission of various by-products into the ocean or air. To limit these activities is create the basis for a legal action brought by some public or private party that would lead the legal authorities to intervene to set aside the let-it-be judgment of the "invisible judge." The theorems developed below apply to the *laissez faire* mechanism. The theorems do not apply to other explicit acts of the legal authorities which nevertheless affect the property system, e.g., enclosures, homestead acts, state-sponsored taking or destruction of property, or various legalizations of conquest.

One might wonder why the description of the *laissez faire* mechanism of appropriation seems to be new in spite of the great simplicity of the mechanism. There seems to be a combination of reasons. One reason is that discussions of appropriation tend to be restricted to a rather mythical state of nature [e.g., Locke 1960 (1690)] or original position in the philosophical literature.

Secondly, the role of appropriation in ordinary production tends to be overlooked due to the common but ill-formed idea that the ownership of the product is part of the ownership of some already-owned capital assets. Hume's notion of "accession" might be included as an example. In any case, this idea is easily defeated by considering the case when the capital assets are rented out rather than the other factors being hired in. In general, it is the party who acquires the contractual position of being the terminal owner of the used-up inputs (e.g., when capital assets are rented, their services are bought and then used up) who has the legally defensible claim on the produced outputs which can then be sold without legal challenge. Thus that party, by virtue of its contractual position (not its asset ownership), legally appropriates the liabilities for the used-up inputs as well as the rights to the produced assets. The net value of those assets and liabilities is the "residual" so it might be said that residual claimancy is a contractual position, not an attribute of owning land or other capital assets.

The idea that "being the firm" is a property right that is owned rather than a contractual position is nevertheless quite common in the economics and legal literature, not to mention in lay beliefs. Entrepreneurs are "bidding for ownership of the firms" [Hirshleifer 1970, 124] and become the "owners of the productive opportunity" [Ibid., 125]. A proprietor may sell "the rights to the transformation function" or "his rights to the venture" [Fama and Jensen 1996, 341] to another proprietor. The entrepreneur is the "owner of a production function" [Haavelmo 1960, 210] and even Robinson Crusoe "owns the firm" [Varian 1984, 225]. But the most common fallacy occurs when the ownership of a corporation (which owns various economic resources) is confused with the contractual role of the corporation. It is only the contractual role (e.g., hiring in labor or hiring out capital), that determines whether a corporation or any other resource owner ends up being the firm (legally appropriating the assets and liabilities created in production) or only a resource supplier to that firm. If one continually thinks that "being the firm" is an owned property right (like "ownership of a production function"), then the market mechanism of appropriation will be entirely overlooked—as indeed seems to have happened.

Property theory also differs markedly from the "law and economics" literature since the latter attempts to reduce legal rules and juridical mechanisms to efficiency considerations such as the minimization of transactions costs. Similarly "the 'new institutional economics,' [consists] in large part of transactions cost analysis of property rights, contracts and organizations."

[Rutherford 2001, 187] In contrast, property theory uses jurisprudential norms central to the subject matter such as the responsibility principle introduced in the next section. The fundamental theorems relate the responsibility principle to the legal mechanism for the *laissez faire* appropriation of the assets and liabilities created in production-consumption activities. By attempting to reduce "law" to "economics," the law-and-economics literature and the new institutional economics neither formulates the responsibility principle nor the market mechanism for legal appropriation.[11]

The Fundamental Theorems with Property Rights

The Responsibility Principle

The development of the concept of "property rights" (previously only implicit in the framework) together with the market mechanism of appropriation allows a richer restatement of the fundamental theorems. The *laissez faire* mechanism of appropriation, like the price mechanism, is an institutional arrangement that can be judged according to certain normative principles. Previously I considered the notion of a socially-possible realization of intentional actions as the normative principle used to evaluate the contractual mechanism.

How can one evaluate the market mechanism of appropriation? The relevant juridical principle is indicated by the example given of the legal authorities intervening to assign or impute the legal liability for destroyed property. The principle was to assign or impute the legal or *de jure* responsibility (i.e., the legal liability) for the destroyed property to the party *de facto* responsible for destroying the property (again, this is Duguit's legal responsibility to "compensate another for damages due to one's own fault"). Applied to both the creation and termination of property rights, the juridical *responsibility principle* is: impute *de jure* (or legal) responsibility in accordance with *de facto* responsibility.[12] Under what circumstances does the market mechanism of appropriation "automatically" (i.e., with no legal intervention) satisfy the responsibility principle?

To relate the responsibility principle to our previous treatment, it suffices to consider the connection between intentional actions and *de facto* responsibility. When people carry out intentional actions, then they are *de facto* responsible for the results of those actions. Previously SIA stood for a socially consistent set of intentional decisions by the parties about production and consumption (which includes intended changes in stocks under our present interpretation). When realized, SIA would stand for *de facto* responsibility. The sign conventions for SIA were chosen so that the positive components represented commodities consumed or used up while the negative components represented commodities produced. Hence –SIA would be a quantity-balanced node assignment with the positive (resp. negative) components at each node representing the assets (resp. liabilities) for which the party represented by the node is *de facto* responsible when the intentional actions are realized.

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[11] For example in the Putterman and Kroszner anthology [1996] of papers on the "economic" nature of the firm, none of the papers pose the question of appropriation in their treatment of the firm. The question of appropriation in the firm is similarly ignored in the "economics of property rights" [e.g., Furubotn and Pejovich 1974] and in the so-called "property rights approach" to the firm [e.g., Hart and Moore 1990; Hart 1995].

[12] This principle can be taken as a modern and symmetrical treatment of the old Lockean [Locke, 1960 (1690)] or natural rights principle [see Schlatter 1951] of people getting the "fruits of their labor." See Ellerman 1992 for more analysis of the responsibility principle.

To summarize the connections between the underlying interpretations: the boundary of voluntary contracts are intentional decisions, and when intentional decisions are realized, then the parties are *de facto* responsible for the results.

The Fundamental Theorems and the Responsibility Principle

Theorem 5 is Theorem 3 restated in terms of responsibility.

Theorem 5: First Fundamental Theorem of Property Theory: If there is a contractual matching, then the market mechanism of appropriation satisfies the responsibility principle.

Proof. Since VC = FT, the realized actions ∂FT are the social intended actions SIA = ∂VC and that common node assignment represents what the parties are *de facto* responsible for consuming (with production listed as negative). The negative of that node assignment gives that for which the parties are *de facto* responsible for producing (with consumption as negative). Thus "assets and liabilities for which the parties *de facto* responsible" = $-\partial$FT = –SIA = $-\partial$VC = LA = "assets and liabilities for which the parties are *de jure* responsible," i.e., the responsibility principle is satisfied. QED

Thus the "natural system of property and contract" operates, as if guided by an "invisible hand," to satisfy the basic juridical principle of imputing legal responsibility in accordance with *de facto* responsibility. When there is a contractual matching, the system operates as an "invisible judge" to correctly impute legal responsibility. The contrapositive of the theorem states that if the responsibility principle was violated within the production-consumption activities of the parties represented by the nodes then it would show up as a mismatch in the transactions represented on the arcs. This is a property-theoretic refutation of Marx's charge that there could be exploitation in the "hidden abode of production" while the sphere of exchange "is in fact a very Eden of the innate rights of man" [Marx 1967, 176].

Theorem 6: Second Fundamental Theorem of Property Theory: Given the proposed *de facto* responsible actions –SIA that satisfy the quantity balance condition Aug(SIA) = 0, there exists a supporting arc assignment T such that if T is accepted by the parties as voluntary contracts VC transferring property rights, then the contractual matching and market mechanism of appropriation would assign legal responsibility (LA) in accordance with the realized *de facto* responsibility (–SIA).

Proof: The existence of T follows directly from Theorem 2 and the rest is immediate from the definitions. By the existence of T accepted as VC, we have ∂VC = SIA, and the matching VC = FT implies that the net inflow realized is ∂FT = ∂VC = SIA. The realized net outflow is –SIA = $-\partial$VC = LA. QED

Conclusion

The first fundamental theorem shows the underlying logic of the natural system of property and contract. The contractual mechanism enforces a contractual matching that keeps the legal ownership correlated with the *de facto* possession and control of commodities. The *laissez faire* mechanism defines the legal appropriation of assets and liabilities in terms of the contracts. Since people will have the first (respectively, last) possession of what they produce (consume), the market-based mechanism will ensure that they legally appropriate the rights to the assets they produce (the liabilities for the assets they consume or use up). Thus Hume's contractual mechanism reaches back to enforce the natural rights principle associated with John Locke.

Appendix 1: Proofs of Theorems 1 and 2

Theorem 1: For any T in $C(G_1, \mathbf{R}^n)$, $\mathrm{Aug}(\partial T) = 0$.

Proof:

$$\mathrm{Aug}(\partial T) = \sum_{i=1}^{I}\left[\sum_{h(e_j)=n_i} T(e_j) - \sum_{t(e_j)=n_i} T(e_j)\right]$$

$$= \sum_{j=1}^{J}\left[T(e_j) - T(e_j)\right] = 0$$

where, in the last term, the positive $T(e_j)$ is from the node at the head of e_j and the negative $T(e_j)$ is from the node at the tail of the arc. QED

Theorem 2: Let S be a node assignment on a connected graph G such that Aug(S) = 0. There exists an arc assignment T on G such that $\partial T = S$, i.e., $\mathrm{Ker(Aug)} \subseteq \mathrm{Im}(\partial)$.

Proof: Let n_m be any node of G which will be called the "market-maker" node. Since the sum of all the S(n) over all the nodes of G is 0,

$$\sum_{n \neq n_m} S(n) = -S(n_m).$$

Hence the 0-chain S can be written as: $\sum_{n \neq n_m} S(n)[n - n_m] = S$. Since G is a connected graph, for any node n different from the market-maker, there is a path ±e±e'±e"...±e*from n_m to n where the coefficients ±1 are determined according to whether the arc is oriented along (+1) or against (−1) the path. Let $T_n = S(n)[\pm e \pm e' \pm e'' \ldots \pm e^*]$. I show that $\partial T_n = S(n)[n - n_m]$ so that the sum of the T_n is the desired T such that $\partial T = S$. I consider the endpoints of the path separately from the interior points. Consider a point n' on the path where, say, $\xrightarrow{e'} (\text{node } n') \xleftarrow{e''}$ and where the arc e' is in the direction of the path. Then the corresponding term S(n)[+e' − e"] in T_n contributes nothing to the boundary since $\partial T_n(n') = \{+S(n) - (+S(n))\}n' = 0$. In a similar manner, the other three possible cases for S(n)[±e'±e"] also have zero boundary. Hence it remains to consider the four cases that arise at the end points e and e* of the path from n_m to n.

$$\partial\{S(n)[+e\pm\ldots+e^*]\} = -S(n)n_m + S(n)n = S(n)[n-n_m]$$
$$\partial\{S(n)[+e\pm\ldots-e^*]\} = -S(n)n_m - (-S(n)n) = S(n)[n-n_m]$$
$$\partial\{S(n)[-e\pm\ldots+e^*]\} = +(-S(n)n_m) + S(n)n = S(n)[n-n_m]$$
$$\partial\{S(n)[-e\pm\ldots-e^*]\} = +(-S(n)n_m) - (-S(n)n) = S(n)[n-n_m]$$

In each case, $\partial T_n = S(n)[n - n_m]$ so $T = \sum_{n\neq n_m} T_n$ yields an arc assignment T such that

$$\partial T = \sum_{n\neq n_m} \partial T_n = \sum_{n\neq n_m} S(n)[n-n_m] = S. \text{ QED}$$

By the proof, we see any party can be selected as the market-maker and the desired bilateral exchanges can be constructed as exchanges of each other party with the market-maker so that the resulting net inflow at the market-maker is equal to the market-maker's intended production-consumption decision—so all the intentions can be realized by the bilateral exchanges.

Appendix 2: Analysis of Externalities and Breaches

A real vector is non-negative if each component is non-negative (zero or positive). A real vector can always be decomposed into the difference of two non-negative vectors. Given two vectors $X = (x_1,\ldots,x_n)$ and $Y = (y_1,\ldots,y_n)$ in $\mathbf{R}^n$, let max(X,Y) be the vector with the maximum of x_i and y_i as its i^{th} component, and let min(X,Y) be the vector with the minimum of x_i and y_i as its i^{th} component. The *positive part* of X is:

$$X^+ = \max(X,0),$$

the maximum of X and the zero vector [the notation 0 is used to denote the zero scalar, zero vector, and the zero arc and node assignments]. The *negative part* of X is:

$$X^- = \max(-X,0) = (-X)^+$$

where also $X^- = -\min(X,0)$. Both the positive and negative parts of X are non-negative (note the slight abuse of language in calling the non-negative vector X^- the "negative part" of X). Every vector X can be represented as the difference of the positive and negative parts, which is called the "Jordan decomposition" in Analysis:

$$X = X^+ - X^-$$

Jordan decomposition.

Two non-negative vectors X and Y are said to be *disjoint* if min(X,Y) = 0, i.e., if each component is zero in one or the other (or both) of the vectors. The positive and negative parts of any vector are disjoint, i.e., $\min(X^+,X^-) = 0$, and their difference is X. All these definitions and results extend immediately to arc and node assignments by applying them to each vector assigned to the arcs or nodes.

There are two types of mismatches between the factual transfers FT and the voluntary contracts (legal transfers) VC. The factual transfers not covered by contracts will be called *externalities*, symbolized EXT, (or "conversions" using more juridical terminology) and the contracts not fulfilled by the factual transfer or delivery of the legally transferred commodities will be called *breaches*, symbolized BRE (i.e., contracts that were breached). The Jordan decomposition of vectors allows precise definitions:

$$\mathrm{EXT} = [\mathrm{FT}^+ - \mathrm{VC}^+]^+ - [\mathrm{FT}^- - \mathrm{VC}^-]^+$$
$$\mathrm{BRE} = [\mathrm{VC}^+ - \mathrm{FT}^+]^+ - [\mathrm{VC}^- - \mathrm{FT}^-]^+.$$

For instance, let us consider commodity vectors that are three-dimensional and, on arc e_j oriented from party A to B, we have $\mathrm{FT}(e_j) = (5, -7, 2)$ and $\mathrm{VC}(e_j) = (3, -9, -3)$.

$$\text{A} \xrightarrow[\mathrm{VC}(e_j) = (3, -9, -3)]{\mathrm{FT}(e_j) = (5, -7, 2)} \text{B}$$

Then $\mathrm{EXT}(e_j) = [(5, 0, 2) - (3, 0, 0)]^+ - [(0, 7, 0) - (0, 9, 3)]^+ = (2, 0, 2)^+ - (0, -2, -3)^+ = (2, 0, 2)$ and similarly, $\mathrm{BRE}(e_j) = (0, -2, -3)$.

$$\text{A} \xrightarrow[\mathrm{BRE}(e_j) = (0, -2, -3)]{\mathrm{EXT}(e_j) = (2, 0, 2)} \text{B}$$

Thus party B converted 2 units of the 1st commodity and 2 units of the 3rd commodity from party A, and party B also breached contracts to transfer 2 units of the 2nd commodity (note that 7 units were fulfilled) and 3 units of the third commodity to party A. The interpretation of the mixed signs in the third component of $\mathrm{FT}(e_j) = (5, -7, 2)$ and $\mathrm{VC}(e_j) = (3, -9, -3)$ is interesting. Not only did B not fulfill the contract to transfer 3 units of the 3rd commodity to A but "added insult to injury" by converting an additional 2 units.

It might be noted that the difference between the factual and legal transfers (FT – VC) is the same as the difference between the externalities and breaches (EXT – BRE), i.e.,

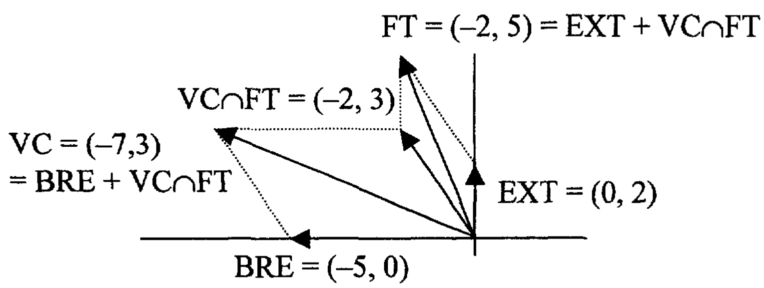
$$
\begin{aligned}
&\mathrm{EXT} - \mathrm{BRE} \\
&= [\mathrm{FT}^+ - \mathrm{VC}^+]^+ - [\mathrm{FT}^- - \mathrm{VC}^-]^+ - \{[\mathrm{VC}^+ - \mathrm{FT}^+]^+ - [\mathrm{VC}^- - \mathrm{FT}^-]^+\} \\
&= [\mathrm{FT}^+ - \mathrm{VC}^+]^+ - [\mathrm{VC}^+ - \mathrm{FT}^+]^+ + [\mathrm{VC}^- - \mathrm{FT}^-]^+ - [\mathrm{FT}^- - \mathrm{VC}^-]^+ \\
&= [\mathrm{FT}^+ - \mathrm{VC}^+]^+ - \{-[\mathrm{FT}^+ - \mathrm{VC}^+]\}^+ + [\mathrm{VC}^- - \mathrm{FT}^-]^+ - \{-[\mathrm{VC}^- - \mathrm{FT}^-]\}^+ \\
&= [\mathrm{FT}^+ - \mathrm{VC}^+]^+ - [\mathrm{FT}^+ - \mathrm{VC}^+]^- + [\mathrm{VC}^- - \mathrm{FT}^-]^+ - [\mathrm{VC}^- - \mathrm{FT}^-]^- \\
&= [\mathrm{FT}^+ - \mathrm{VC}^+] + [\mathrm{VC}^- - \mathrm{FT}^-] \\
&= [\mathrm{FT}^+ - \mathrm{FT}^-] - [\mathrm{VC}^+ - \mathrm{VC}^-] \\
&= \mathrm{FT} - \mathrm{VC}.
\end{aligned}
$$

In the example, $EXT(e_j) - BRE(e_j) = (2, 0, 2) - (0, -2, -3) = (2, 2, 5) = (5, -7, 2) - (3, -9, -3) = FT(e_j) - VC(e_j)$.

This also means that there is a common overlap of FT and VC that could be thought of either as the fulfilled contracts or as the transfers covered by contracts. Moreover, that overlap is the factual transfers except for the externalities and it is also the voluntary contracts except for the breaches, i.e., "Overlap" = FT – EXT = VC – BRE. That overlap is the natural definition of the *intersection* of FT and VC:

$$VC \cap FT = VC - BRE = FT - EXT.$$

To illustrate graphically with two-dimensional vectors, let $FT(e_j) = (-2, 5)$ and let $VC(e_j) = (-7, 3)$. Then $EXT(e_j) = (0, 2)$ and $BRE(e_j) = (-5, 0)$ and the overlap or intersection is $VC \cap FT(e_j) = (-2, 3)$. Dropping the reference to the arc e_j, this is illustrated as follows.



Graph 1: Graphical Illustration of FT = EXT + VC∩FT and VC = BRE + VC∩FT

Since FT – EXT = VC – BRE, when EXT = 0 and BRE = 0, then there is a contractual matching, FT = VC. Conversely when there is a contractual matching, FT = VC, then clearly $FT^+ = VC^+$ and $FT^- = VC^-$ so EXT = 0 and BRE = 0. These results are summarized as:

Theorem 7: There is a contractual matching (FT = VC) if and only if there are no externalities (EXT = 0) and no breaches of contracts (BRE = 0).

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